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NYU-MWP Memo. No. 11 + Comments

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Expectation Value for the Size of Devastation Areas Produced by N Bombs.

The Problem

The following experiment is defined. n circles of constant radius r are drawn at random and independently in the x,y-plane. The distribution function for the center (x,y) of any one of the circles is p(x,y); thus

$$\iint_A p(x,y)\,dx\,dy$$

is the probability that the center will lie in a set A. For a given set of n such circles, let X be the area of the set of points within at least one of the n circles. The calculation of the distribution function of X seems to be very difficult. Instead, we propose to find merely the expected value of X, denoted here by E_n.

Solution

We define

$$f(a,b) = \iint_{C(a,b)} p(x,y)\,dx\,dy,$$

where C(a,b) is the circle of radius r and with center (a,b). Then it is asserted that

$$E_n = \iint_{-\infty}^{\infty} \left\{1 - (1 - f(x,y))^n\right\} dx\,dy.$$

The proof of this statement may be left to the reader, or will be supplied on request.

A SPECIAL CASE

Suppose $p(x,y)$ is a "flat" distribution function,

$$p(x,y) = \begin{cases} 1/B & \text{for } (x,y) \text{ within a given domain D of area B} \\ 0 & \text{elsewhere.} \end{cases}$$

Let $a = \pi r^2$ be the area of each circle. Then if a is sufficiently small we can set, without serious error,

$$f(x,y) \simeq \begin{cases} a/B & \text{for } (x,y) \text{ within the domain D} \\ 0 & \text{elsewhere.} \end{cases}$$

From (1) it follows that

$$(2) \qquad E_n \simeq B[1 - (1 - \tfrac{a}{B})^n].$$

In order that

$$E_n \simeq (1 - \alpha)B$$

we must have

$$(3) \qquad n \simeq \log \alpha / \log(1 - \tfrac{a}{B}).$$

Thus if $\alpha = .05$ and $a/B = .01$,

$$n \simeq 300.$$

THE GENERAL CASE

Since

$$\iint_{-\infty}^{\infty} f(x,y)dx\ dy = a,$$

it follows from (1) that

$$(4) \qquad E_n = na - \sum_{i=2}^{n}(-1)^i\ C(n,i) \sum_{-\infty}^{\infty} f^i(x,y)\ dx\ dy.$$

For "small" a and n we can make the approximation

$$(5) \quad \iint_{-\infty}^{\infty} f^i(x,y)\ dx\ dy \simeq a^i \iint_{-\infty}^{\infty} p^i(x,y)\ dx\ dy.$$

Thus we obtain the approximate formula

$$(6) \quad E_n \simeq na - \sum_{i=2}^{n}(-1)^i\ C(n,i)\ a^i \iint_{-\infty}^{\infty} p^i(x,y)\ dx\ dy.$$

For example, if

$$p(x,y) = 1/(2\pi)\ e^{\frac{-(x^2+y^2)}{2}},$$

then

$$\iint_{-\infty}^{\infty} p^i(x,y)\ dx\ dy = \frac{1}{i(2\pi)^{i-1}}$$

Thus

$$(7) \qquad E_n \simeq na - \sum_{i=2}^{n} (-1)^i\ C(n,i)\ a^i \frac{1}{i(2\pi)^{i-1}}$$

$$= na[1 - \frac{1}{n} \sum_{i=2}^{n} (-1)^i \frac{C(n,i)}{i} (\frac{a}{2\pi})^{i-1}].$$

APPENDIX. THE VARIANCE IN THE "FLAT" CASE.

By definition of variance,

$$(8) \qquad \sigma_n^2(X) = \mathcal{E}(X - \mathcal{E}(X))^2 = \mathcal{E}(X^2) - \mathcal{E}^2(X)$$

$$= \mathcal{E}(X^2) - E_n^2 \geq 0,$$

where $\mathcal{E}(f(X))$ denotes the expected value of $f(X)$. But since $X^2 \leq B^2$, it follows that

$$E_n^2 \leq \mathcal{E}(X^2) \leq B^2 .$$

Substituting from (2) we have

$$B[1 - (1 - \frac{a}{B})^n] \leq \mathcal{E}(X^2) \leq B^2.$$

Hence from (8)

$$(9) \qquad 0 \leq \sigma_n^2 \leq B^2(1 - \frac{a}{B})^n (2 - (1 - \frac{a}{B})^n) .$$

Set $\alpha = (1 - \frac{a}{B})^n$. Then as $n \to \infty$, $\alpha \to 0$, and

$$(10) \qquad E_n = B(1 - \alpha) \to B$$

$$\sigma_n^2 \leq B^2 \alpha (2 - \alpha) \to 0.$$

COMMENTS ON NYU - AMG Memo. No. 11.

The enclosed Memo No. 11 constitutes a copy of a letter received from Lieut. (j.g.) Herbert Robbins, Post Graduate School, Annapolis.

The mathematical problem is concerned with the following question: Suppose we want to devastate a given area with N bombs, each bomb having the same radius of destruction r and each being aimed at a specific target. What is the expectation value for the total devastated area? This expected total area will, of course, depend upon the N distribution functions $p_\nu(x,y)$ $(\nu = 1,2,\ldots,N)$, where $p_\nu(x,y)$ is the probability density that the ν th bomb hits the point (x,y).

The probability that a given point (a,b) will not be affected by the νth bomb is equal to the probability that this bomb will hit outside a circle of radius r about the point (a,b). This probability is

$$1 - f_\nu(a,b) \ ,$$

where

$$f_\nu(a,b) = \iint_{C(a,b)} p_\nu(x,y)\, dx\, dy,$$

$C(a,b)$ being the circle of radius r about (a,b).

The probability that the point (a,b) will not be affected by any of the N bombs is, therefore, the product

$$\prod_{\nu=1}^{N} [1 - f_\nu(a,b)] = [1 - f_1(a,b)]\ldots.[1 - f_N(a,b)].$$

The probability that (a,b) will be affected by at least one bomb is then

$$1 - \prod_{\gamma=1}^{N} [1 - f_\gamma(a,b)].$$

Thus we arrive at the following value for the total area which we might expect to devastate with the N bombs

$$E_N = \iint_{-\infty}^{\infty} \left\{ 1 - \prod_{\gamma=1}^{N} [1 - f_\gamma(a,b)] \right\} da\, db.$$

In Robbins' letter it is assumed that all the distribution functions $p_\gamma(x,y)$ are the same. This would mean that either all the bombs seek the same target or that they are all dropped blindly over a certain area. The formula in this comment is given for an individual pattern of targets. Then a reasonable assumption would be that p_γ is a Gauss distribution about the target point (x_γ, y_γ). According to the selection of the target pattern we obtain smaller or larger expectation values E_N.

Of course, this is nothing but a framework which would have to be filled by numerical calculations under specific assumptions. Probably similar considerations are familiar to the people concerned with bombing (including three dimensional bombing). If this should not be the case, then it might be worthwhile to proceed in the mathematical analysis of expectation values in order to find reasonable guidance for numerical work on economic bombing patterns.

Since, in general, it is easier to deal analytically with expectation values rather than with probabilities, it might be worthwhile to consider expectation values of area also in other problems of an operational character such as Search problems or Patrol problems.

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